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A FORMULA FOR CUBIC MEASUREMENT ON A PLANE

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ABSTRACT

Formulas are derived to calculate with approximated approximation, formulas that allow numerical evaluation of double integrals and triple integrals on flat two-dimensional surfaces, by means of measurements of the integrating function values, taken in a grid that is plotted in the plane. The method is analogous to that used to derive Simpson's formula for a simple integral.

KEYWORDS: Numerical Analysis; Numerical integration; Integral calculus.

UNA FÓRMULA DE CUBICACIÓN SOBRE EL PLANO

RESUMEN

Se deducen fórmulas para calcular con aproximación acotada, fórmulas que permiten valorar numéricamente integrales dobles e integrales triples sobre superficies bidimensionales planas, mediante mediciones de los valores de la función integrando, tomadas en una retícula cuadriculada que se trace en el plano. El método es análogo al que se usa para deducir la fórmula de Simpson para una integral simple.

PALABRAS CLAVE: Análisis Numérico; Integración numérica; Cálculo Integral.

UMA FÓRMULA DE CUBICAÇÃO NO PLANO

RESUMO

As fórmulas são derivadas para calcular com aproximação aproximada, fórmulas que permitem a avaliação numérica de integrais duplas e integrais triplas em superfícies bidimensionais planas, por meio de medições dos valores de função de integração, tomadas em uma grade que é plotada no plano. O método é análogo ao usado para derivar a fórmula de Simpson para uma integral simples.

PALAVRAS-CHAVE: Análise Numérica; Integração numérica; Cálculo integral.

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1. Those who study integral calculus or any of its numerous applications know the definite integration formula known as Simpson's Rule, which is used to approximately compute a definite integral in the case of a function that is described by its values in equidistant points. As you remember, the rule is

$$\int_{xo}^{xn} f(x) \cdot dx = (y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots 4y_{n-1} + y_n) \times (x_n - x_o)/3n$$

where the symbols have the meaning shown in Drawing 1. It is worth remembering two characteristic features of this formula:

1. The number of equidistant values used is: o 3, o 5, o 7,..., or, in general, an odd number in the form of n + 1
2. The sum of the coefficients of said terms is $3n$
3. The number of intervals between two consecutives (y_i, y_{i+1}) is n and is even
4. The sum of said intervals is $x_n - x_o$

2. It is surprising that in the didactic mathematical literature there is almost no mention of the evident fact that just as one deduces Simpson's quadrature formula, so can one deduce a formula for volume measurement to calculate (even approximately) the value of double integrals defined on a region R of the plane, in two variables, that is, integrals of the form

$$\iint_R f(x, y)\, dx \cdot dy \qquad (2.1)$$

An integral in this form expresses the volume contained between the surface and the plane determined by the Cartesian axes OX, OY, which are perpendicular to each other and perpendicular to OZ. For this reason, such a formula can have numerous applications. Examples of such applications would be:

a. To calculate the volume held in a reservoir knowing a sufficiently refined depth grid

b. To calculate the mineral content of a hill, knowing its horizontal grid and mineral tenors for each probe up to a determined depth

It is worth noting that in mathematics textbooks commonly used in our schools, in general, the topic of formulas for the approximation of volume is rarely mentioned, even in good texts about numerical analysis. The purpose of this note is to deduce a formula for volume measurement "in the manner of Simpson" and to show how it could be used.

3. First: consider the problem of calculating a volume between a quadric surface in two variables and the plane of the two axes OX and OY.

Let this be the quadric surface

$$z(x, y) = a + b_1x + b_2x^2 + c_1y + c_2y^2 + dxy \quad (3.1)$$

Where a, b_1, b_2, c_1, c_2, d are real constants such that the coefficient of the second degree terms are not null, meaning such that

$$b_2^2 + c_2^2 + d^2 > 0 \qquad (3.2)$$

Consider on the other hand, the region of plane OXY formed by the R chart shown in Fig. 2, and whose vertices are the four points (1,0), (0,1), (-1,0) and (0,-1). Its surface measures, as is evident, 2 units of area. Calculating the function $z(x, y)$ in the center of the square and its four corners, one obtains

$$z(0,0) = z_0 = a$$
$$z(1,0) = z_1 = a + b_1 + b_2$$
$$z(0,1) = z_2 = a + c_1 + c_2$$
$$z(-1,0) = z_3 = a - b_1 + b_2$$
$$z(0,-1) = z_4 = a - c_1 + c_2$$

Inverting this system, one obtains the coefficients a, b_1, b_2, c_1, c_2, in terms of the dimensions z_0, z_1, z_2, z_{3}, $_4$ of the surface (x, y) on the plane OXY:

$$a = z_0$$
$$b_1 = (1/2)\,(z_1 - z_3)$$
$$b_2 = (1/2)\,(z_1 + z_3) - z_0$$
$$c_1 = (1/2)\,(z_2 - z_4)$$
$$c_2 = (1/2)\,(z_2 + z_4) - z_0$$

Thus, the five coefficients are determined by the values of $z(x, y)$ on the five points indicated in the square R, and the equation of the quadric surface can be written as a determinant as

$$\begin{vmatrix} 1 & x & x^2 & y & y^2 & z \\ 1 & 0 & 0 & 0 & 0 & z_0 \\ 1 & 1 & 1 & 0 & 0 & z_1 \\ 1 & 0 & 0 & 1 & 1 & z_2 \\ 1 & -1 & 1 & 0 & 0 & z_3 \\ 1 & 0 & 0 & -1 & 1 & z_4 \end{vmatrix} = 0$$

Figure 1

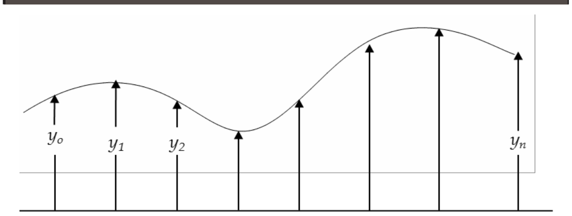


Figure 2

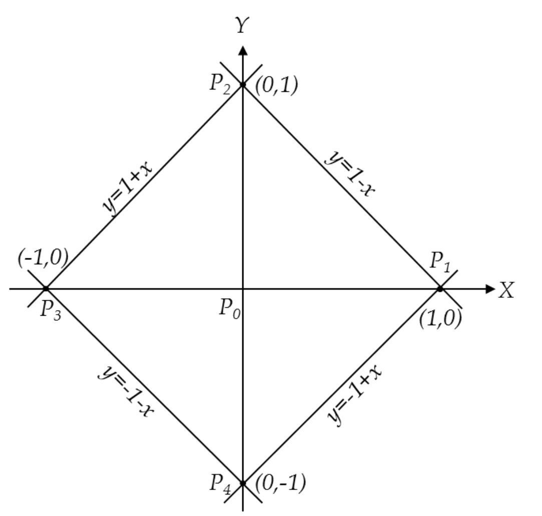
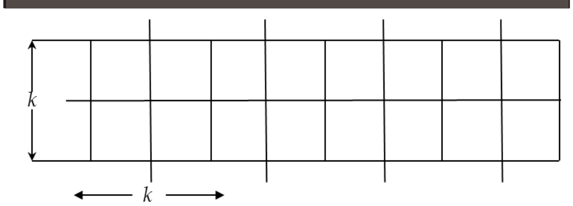


Figure 3



4. The volume that covers $z(x, y)$ over region R (whose area measures 2 units) is the integral

$$V = \iint_R z(u, v)\, du \cdot dv =$$

$$= \iint_R (a + b_1 u + b_2 u^2 + c_1 v + c_2 v^2) du \cdot dv \qquad (4.1)$$

The value of these integrals can be calculated term to term:

$$\iint_R du \cdot dv = 2$$

$$\iint_R u{\cdot}du{\cdot}dv = \int_{-1}^{0} \int_{v=-(1+u)}^{v=1+u} u{\cdot}du{\cdot}dv + \int_{0}^{1} \int_{v=-(1-u)}^{v=1-u} u{\cdot}du{\cdot}dv$$

$$\iint_R u^2 \cdot du \cdot dv = 1/3$$

$$\iint_R v \cdot du \cdot dv = 0$$

$$\iint_R v^2 \cdot du \cdot dv = 1/3$$

$$\iint_R u \cdot v \cdot du \cdot dv = 0$$

Consequently, one obtains

$$V = 2a + b_2/3 + c_2/3 \qquad (4.2)$$

And substituting the values of the coefficients, we have:

$$V = [(4/3)]\, z_0 + (1/6)\,(z_1 + z_2 + z_3 + z_4) \qquad (4.3)$$

By way of verification of this formula it can be seen that if the surface that covers R is the plane $z(x, y) = 1$, the volume in question is a parallelepiped of base R with an area equal to 2 and height equal to 1, whose volume is 2 cubic units. This is precisely what is stated in the formula below:

$$V = [(4/3) \times 1 + (1/6)\,(1 + 1 + 1 + 1)] = 12/6 = 2 \text{ (cubic units)}$$

Figure 4

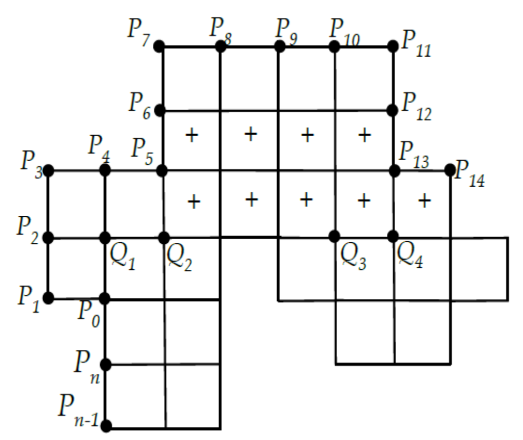
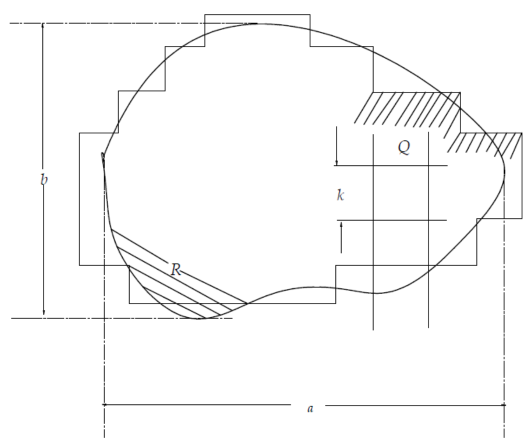


Figure 5



This formula (4.3) expresses the volume sought as a linear combination of the five dimensions z_o, z_1, z_2, z_3 y z_4, in which it should be noted that the sum of the five coefficients is equal to 2, as seen above.

5. Considering that there is a rectangle formed by two squares on side k joined by one of its sides, as in **Figure 3**, in plane OXY, and that on it, in the space, is the quadric surface

$$z(x, y) = a + b, x + b_2x^2 + c_1y + c_2y^2 + dxy$$

accompanied by the condition (3.2), we see that the volume of the quadric above the rectangle is obtained by applying formula (4.3) to the two squares and adding. Thus, one obtains

$$V = [(4/3)(z_{11} + z_{31} + (1/6)(z_{00} + z_{02} + z_{40} + z_{42}) + (1/3)(z_{20} + z_{22})]\, k^2/2 \qquad (5.1)$$

Where $z_{hk} = z(x_h, y_k)$ is the dimension of the quadric at the coordinate point x_h, y_k.

6. Juxtaposing one square to the other, each pair of them joined by one of their sides, a figure such as that shown in **Figure 4** can be formed. The perimeter of this figure is a polygon $P_0P_1P_2 \cdots P_{n-1}P_n$ whose successive points are equidistant. For some of these points the two adjacent segments at the point are collinear, such is the case of P_2, P_4, P_6, P_8, P_9, etc. We will call these points flat nodes.

At some other points of the polygon the two adjacent segments form a 90° angle with respect to the interior of the polygon as occurs in P_1, P_3, P_7, P_{11}, P_{14}, etc. We will call these points convex nodes. And at points like P_0, P_5, P_{13}, $\cdots$, where an angle of 270° is formed with respect to the interior of the polygon, we will call them concave nodes. If p is the number of concave nodes, it easily demonstrates that the number of convex nodes is $p + 4$. Each flat point belongs to two squares, each convex corner belongs to a square, and each concave corner belongs to three squares. At points like Q_1, Q_2, etc. that belong to four squares, we will call them internal nodes. The centers of the squares will be called C_1, C_2, $\cdots$, etc.

7. The volume that a quadric surface

$$z(x, y) = a_0 + b_1x + b_2x^2 + c_1y + c_2y^2$$

covers above the shaded polygonal area of Fig. 5 can be calculated applying Formula (4.3) at each of the squares that form the polygonal area and adding above all of them.

The result is almost evident and can be written

$$V = [(4/3)S_0 + (1/6)\, S_1 + (1/3)\, S_2 + (1/2)\, S_3 + (2/3)\, S_4]k\ 2/2 \qquad (5.1)$$

where:

S_0: Sum of the dimensions $z(x, y)$ in the centers of the squares

S_1: Sum of the dimensions $z(x, y)$ in the convex nodes

S_2: Sum of the dimensions $z(x, y)$ in the flat nodes

S_3: Sum of the dimensions $z(x, y)$ in the concave nodes

S_4: Sum of the dimensions $z(x, y)$ in the internal nodes

It is interesting to note some numerical relationships that must meet the numbers of the addends of these sums. To express them, we can call them

N_0: Number of squares (or centers of squares)

N_1: Number of convex nodes

N_2: Number of flat nodes

N_3: Number of concave nodes

N_4: Number of internal nodes

The N_0 squares considered one by one have $4 \cdot N_0$ vertices (or nodes), so that counting each convex node one time, each flat node two times, each concave node three times, and each internal node four times, we should have:

$$4\,N_0 = N_1 + 2N_2 + 3N_3 + 4N_4 \qquad (8.1)$$

and since it has already been noted that

$$N_1 = N_3 + 4 \qquad (8.2)$$

it can be deduced that

$$2(N_0 - N_4 - N_3 - 1) = N_2$$

which indicates that the number of flat nodes should be even, and that

$$N_0 = N_4 + N_3 + N_2/2 + 1 \qquad (8.3)$$

On the other hand, it was already seen that upon summing the coefficients of the dimensions of each square (Formula 4.3) we get the number 2, so by summing the N_0 squares we should get $2 \cdot N_0$, and the sum of coefficients in Formula 7.1 will be

$$(4/3)N_0 + (1/6)N_1 + (1/3)N_2 + (2/3)N_4 = 2N_0$$

where

$$(2/3)N_0 + (1/6)\,(N_3 + 4) + (1/3)N_2 + (1/3)N_3 + (2/3)\,N_4$$

Dividing this equation by 2 results again in **Equation 8.3**.

With this background, it is now possible to write a formula to calculate with adjustable approximation an integral of the form

$$\iint_R f(x, y)\, dx \cdot dy$$

R being a simple connected region of the plane OXY and $f(x, y)$ being a bounded, continuous, and summable function on R.

The procedure for valuing this integral is as follows:

a. Admit that $f(x, y)$ is approximate by a quadric surface $z(x, y)$ like (3.1), in the entire region R, in the sense that for each point (x, y) of R we have to

$$|f(x, y) - z(x, y)| < \varepsilon$$

ε being a prescribed positive number and as close to zero (0) as needed to refine the results and agree with the size of the grid that immediately follows.

b. Draw a grid of caliber k that covers all of R, and such that the variation of $f(x, y)$ inside any frame never exceeds ε.

c. Choose a closed polygon in the grid that does not move away at a greater distance than k in the contour of R, which is a simple, rectifiable, and closed Jordan curve, of length L. This polygon is called an approximate polygon, and the region it encloses is shown with Q. It is simple to demonstrate that if k tends to zero, the area of Q tends, as a limit, to the area of R, and the perimeter of Q tends to the perimeter of R.

d. Indicate the centers and the corners of all the squares that are enclosed by the aforementioned polygonal. Said corners are the nodes of the polygonal network.

e. Classify the set of M nodes of the polygonal network in the four disjointed subsets:

M_1: subset of convex nodes

M_2: subset of flat nodes

M_3: subset of concave nodes

M_4: subset of internal nodes

Thus, we have the partition

$M = M_1 \cup M_2 \cup M_3 \cup M_4$, with $M_i \cap M_j = \emptyset$ for $i \neq j$

f. Calculate $f(x, y)$ in each of the nodes in set M, whose number is $N_1 + N_2 + N_3 + N_4$.

Alternatively, $f(x, y)$ can be given in a numeric table or drawn as a family of contour lines on plane OXY.

g. Calculate numerically or algebraically (or read, or observe graphically, or measure physically on a scale model) the value of $f(x, y)$ on each of the centers of the N_0 squares that are enclosed by the approximate polygon. These centers form the subset M_0.

h. Form each of the five summations

$$S_n = \sum_{P_i \in M_n} f(P_i) \quad \text{for } n = 0,1,2,3,4$$

and numerically calculate them.

i. Calculate Expression (7.1)

$$V = [4 \cdot S_0/3 + S_1/6 + S_2/3 + S_3/2 + 2 \cdot S_4/3]\, k^2/2$$

which would give the volume covered by a continuous surface that in each square of the grid coincides with the quadric above the area embraced by the polygon. In the first approximation, it is possible to take this value as an estimate of the proposed integral (2.1), but taking into account that we are admitting errors of approximation by two concepts:

i. By approximating region R with the polygon Q, and

ii. By approximating the function $f(P)$ by the "patches" of quadrics $z(P)$ that coincide with $f(P)$ in the nodes of the grid, but no necessarily on the rest of the points in region R.

10. In order to assess the magnitude of the error of approximation, first observe that region R is the union of Q with $R \cap \bar{Q}$ excluding the points of Q that are not R from the last set of points, that is:

$$R = Q \cup (R \cap \bar{Q} / (Q \cap \bar{Q})$$

where the slash (/) is the difference sign between sets, and the slash above a set indicates its complement. So

$$\iint_R f(P)dP = \iint_Q f(P) \cdot dP + \iint_{R/Q} f(P)dP - \iint_{Q/R} f(P)dP \tag{10.1}$$

The first integral on the right side of this equation is

$$\iint_Q f(P) \cdot dP + \iint_Q z(P) \cdot dP + \iint_Q [f(P) - z(P)]dP = V + \iint (f - z) \cdot dP \tag{10.2}$$

The third term of the right-hand side of **Equation 10.1** which is

$$\iint_{Q/R} f(P) \cdot dP$$

may not be defined by the problem to be solved, or may be null, since the domain where the volume V is sought is R. And by previous consideration, it follows that

$$\lim_{k \to 0} \iint_R dx \cdot dy = \iint_Q dx \cdot dy \Rightarrow \lim \iint_{Q/R} f(P) \cdot dP = 0$$

Substituting (10.2) for (10.1) we thus have

$$\iint_R f \cdot dP - V = \iint_{Q \cap R} (f - z) \cdot dP + \iint_{R/Q} f(P) \cdot dP$$

From here we get the volume calculation error

$$\varepsilon = \left| \iint f(P) \cdot dP - V \right| \leq \left| \iint_{Q \cap R} (f - z)\, dP + \iint_{R/Q} f \cdot dP \right|$$

$$\leq \iint_{Q\cap R} |f-z|\, dP + \iint_{R/Q} |f|\, dP$$

$$\leq \max_{Q\cap R} |f-z| \cdot \text{area } (Q\cap R) + \max_{\frac{R}{Q}} \ |f| \cdot \text{area of } (R/Q)$$

But:

$$\text{area of } Q\cap R \leq \text{area of } R = S$$

$$\text{area of } R/Q \leq \text{perimeter of the border of } R \times k = L\cdot k$$

and calling

$$m = \max_{Q\cap R} |f-z| \leq \max_{R} |f-z| = m^*$$

$$M = \max_{R/Q} |f| \leq \max_{R} |f| = M^*$$

We deduce a dimensioning for the error ε

$$\varepsilon = m\cdot S + MLk \leq m^* \cdot S + M^* \cdot L\cdot k \qquad (10.3)$$

This dimensioning allows us to numerically estimate a maximum limit of the error module of the approximation. In effect, S and L are known data from the region R; k has been chosen at will; m and M were obtained by inspection or another known method of carrying functions to the extreme value.

11. It would now be desirable to be able to prove that the error is infinitesimal with k. In the inequalities (10.3) the factors M, L, k and M^* are evidently infinitesimal with k. For this consideration, it would be necessary and sufficient to show that m (or m^*) tends to zero if $k \to 0$. But this issue of error in the polynomial function setting will be dealt with in another document.

12. In summary, we can state the following formula of approximate numerical integration in two variables and referring to **Figure 5**:

$$\iint_R f(x,y)dx\cdot dy = \left[\frac{4}{3}S_0 + \frac{1}{6}S_1 + \frac{1}{3}S_2 + \frac{1}{2}S_3 + \frac{2}{2}S_4\right] k^2/2 + \varepsilon \qquad (12.1)$$

where S_0, S_1, S_2, S_3, S_4 have the meaning explained in number 7 (Formula 7.1), and

$$|\varepsilon| \leq m\,S + M\cdot L\cdot k$$

whose symbols were already defined in the previous number.

Formula (12.1) might well be called "Simpson's formula on the plane".

13. Among the numerous applications that can be done through this formula it is worth mentioning the following:

- The measurement of the volume of a reservoir or lake by means of probes done along points of the surface that form a well-oriented, well-calibrated, and well-abscised grid
- The volumetry of a coal mantle by drilling holes from the surface and using the grid technique already described
- The measurement of volume for a gravel hill, above the different reference dimensions

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